論　文

Detecting Structural Changes in Stochastic Differential Equation System Based Upon a Bayesian Approach

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Abstract

In recent years, researches have been carried out to investigate the evolving of the Stochastic Differential Equations (SDEs), which are utilized for describing many scientific phenomena, such as, the growth of population, the consumption of natural gas, the behavior of stock prices and returns.

One of the most famous models in finance is to use a SDE, the Geometric Brownian Motion (GBM) to depict the ups and downs of the prices (returns). Furthermore, some studies suggested jump factor should be plugged into the above GBM model, Merton's jump model, for example.

This paper deals with the detecting of structural changes in such SDEs. From the previous studies, we know that there are several parameters in the above mentioned SDEs which determine the evolving values, for instance, drift, diffusion, and jump term etc. The structural changes will occur while the parameters in the SDE change. Equivalently, the evolution of SDE will differ as soon as change points of parameters present.

So far, many studies on detecting structural changes have been carried out, though, the most of them need the assumption of normality. However, for change point or structural change problem, normality seldom holds in many cases. In this paper, we propose a Bayesian approach to detect where a structural change exactly occurs. Our proposed approach is to firstly consider the change sizes of the evolving values (such as prices, returns), secondly to deal with these counting numbers of the change sizes during the fixed intervals as a poisson process, and then by using a Gibbs sampler to detect where the change point lies at. Through our numerical analysis, we find out that our proposed approach works well.

1 Introduction

In recent decades, change point issues have been widely studied under the circumstances of

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different backgrounds, ranging from scientific fields such as DNA identification to social researches, such as the consumption of natural gas, and other social sciences[1][2][3][4][5][6][7][8][9][10][11][12].

However, many studies on how to locate change points have set normality forth as a premise[3][4][5][6][7][8]. One of the important applications of structural change analysis is to detect the change points in the parameters of some functions, such as Stochastic Differential Equations (SDEs), which are usually to be used to model the behavior of stock prices and returns. Wherever a structural change occurs, the evolving values of a SDE will differ. Thus, to detect a structural change plays an important role in risk management, since it makes one adjust the weights of the selected assets of a portfolio, or re-evaluating the Value at Risk (VaR), the risk measurement without delay [10][11][13].

Usually a SDE can be presented as follows.

$$dS = a(S,t)dt + b(S,t)dW \tag{1}$$

where $a(S, t)$ is the trend drift, and $b(S, t)$ is the diffusion drift. On the other hand, some studies suggested jump factor should be plugged into the Equation (1).

$$dS = a(S,t)dt + b(S,t)dW + J(S,t)dq \tag{2}$$

where $J(S, t)dq$ is a jump term, and dq follows a compound Poisson process, $dq = 1$ with probability λdt, and $dq = 0$ with probability $1 - \lambda dt$.

Meanwhile, J(S,t) is a random variable for jump magnitude, usually it follows a special distribution, such as, normal, lognormal, and uniform distributions.

Compared with the business reality, it has been confirmed that SDE (2) behaves more precisely than SDE (1) corresponding to the structural change, or in presence of sudden changes of market.

Usually market information is asymmetric, and market is driven by the market players' own predicts or thoughts, thus, some unexpected jumps (ups and downs) occur naturally. Moreover, the outcomes or the fundamentals of listed firms also affect the market unpredictably, for instance, the rare event of the collapse of Lehman Brothers.

As to detect structural changes in the evolution of a SDE, we consider three situations as follows.

1) Structural changes occur in the trend drift function $a(S, t)$

2) Structural changes occur in the diffusion function $b(S, t)$

3) Structural changes occur in the jump term $J(S, t)$

So far, many studies have focused on Situation 1) Structural changes of the drift function. Thus, in this paper, we mainly consider the detection of the Situations of structural changes 2) and 3).

For Situation 2), and Situation 3), we propose a Bayesian approach to deal with them as follows.

First, we take account for the numbers of the change sizes of a time series of stock prices (returns) in the whole time span, as well as to measure the magnitudes of earthquakes, regarding those observed change sizes (over certain pre-defined levels) as abnormal events, which follow some poisson processes.

Second, we adopt a Bayesian approach, in which some distributional parameters are designed to have their priors, as to catch the change of the intensity of a poisson process, detecting where a change point exactly lies at.

In our numerical examples, we show the differences of quantile distributions and volatilities between the structural changes. And these numerical studies show that our proposed method works well and can more precisely catch and evaluate a structural change.

The rest of this paper is organized as follows. Section 2 reviews and gives rise to some discussions on the Stochastic Differential Equations (SDEs), focusing on a SDE with a Jump factor. Section 3 describes our proposed approach. Section 4 presents some numerical results based upon our proposed approach. Section 5 shows some concluding remarks.

2 Review of the Jump-Diffusion SDE

We simply summarize some statistical properties of the jump-diffusion SDE in this section. In mathematical finance, one typical setting of a SDE is presented as follows.

$$dS = a(S,t)dt + b(S,t)dW \tag{3}$$

where $a(S, t)$ is the drift term, and $b(S, t)$ is the diffusion term, and W is Wiener process. Denoting $a(S, t) = \mu$ and $b(S, t) = \sigma$, we then get the Geometric Brownian Motion (GBM), the most popular SDE in mathematical finance.

$$dS = \mu dt + \sigma dW \tag{4}$$

The simulated 5 paths of Geometric Brownian Motion are shown in Figure 1 with $\mu = 0.03$, $\sigma = 0.2$ and $S_0 = 1$.

It means that dS follows the normal distribution $N(\mu dt, \sigma^2 dt)$. It also has an explicit solution,

$$S_t = S_0 exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}z\} \tag{5}$$

where S_0, S_t are corresponding to the prices at time 0 and t.

Besides, some studies suggested jump factor should be plugged into Equation (3) or (4).

$$dS = a(S,t)dt + b(S,t)dW + J(S,t)dq \tag{6}$$

where $J(S, t)dq$ is a jump term, and dq follows a compound Poisson process, $dq = 1$ with probability λ dt, and $dq = 0$ with probability $1 - \lambda$ dt.

Meanwhile, J(S,t) is a random variable for jump magnitude, usually it follows a special distribution, such as, normal, lognormal, and uniform distributions.

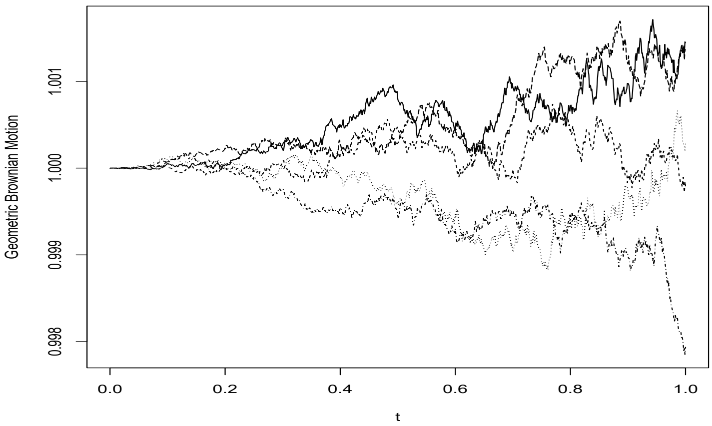

Figure 1 : Simulated paths of Brownian Motion

Equation (6) can actually generates more rare events, just as observed in the tail distribution of S. An extended GMB model is Merton's Jump-diffusion model as follows.

$$dS = \mu dt + \sigma dW + Jdq \tag{7}$$

by adding the jump term, Merton model can be used to explain the phenomena of the heavy (fat) tails of stock returns.

Actually, the solution of (7) can be shown as

$$S_t = S_0 exp\{(\mu - \frac{1}{2}\sigma^2)t + W_t\} \prod_{i=1}^{n} Y_i \tag{8}$$

where n is the number of jumps over the time interval 0 to t, and Y_i is the jump size, and $\prod_{i=1}^{n} Y_i = 1$ if $n = 0$.

Thus, to detect a structural change is to locate where a parameter presents a change point. It is also very crucial to risk management. The behavior of the SDE would change largely as soon as a structural change occurs. The linkage between the VaR and the structural change in the correspondent SDE cannot be neglected.

Let us consider the risk of a portfolio. By denoting

$$E(R_p) = \mu_p = \sum_{i=1}^{n} w_i \mu_i \tag{9}$$

where $E(R_i) = \mu_i$, and w_i is the investing weight of asset i, then the variance of a portfolio return turns out to be

$$\sigma_p^2 = V(R_p) = \sum_{i=1}^{n} {w_i}^2 \sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j<i}^{n} w_i w_j \sigma_{ij} = W^T \Sigma W \tag{10}$$

Hence, a distribution with a different variance yields a different valuation of the portfolio risk. It indicates that a structural change occurred in the parameters of the corresponding SDE will cause a different optimal portfolio components.

Steps of simulating a path of SDE

1) Simulate the times when the jump events occur, these times follow a exponential distribution.

2) Simulate the Brownian path between the jump times.

3) Simulate the Jump sizes for the jump events.

Remarks

Thus far, many studies concerned about a change point detection based upon the some special distribution assumptions, such as normal distribution. However, many researches have not included the jump factor[3][4][5][6][7][8].

As to detect a structural change or a evolution of parameters in a SDE, we hereby propose a Bayesian approach to identify where a structural change lies at. As shown in our previous studies[10][11][12], we count the numbers of the change sizes of the evolving values of a SDE over fixed intervals, and to deal with those change sizes as a compound Poisson process.

3 Bayesian setup

In this section we present our proposed Bayesian approach to detect a structural change, when a change point of the parameters occurs in a SDE. First of all, we define the change sizes of realized values of a SDE. Next, we give a detailed description of our proposed approach. So far, many studies related to detecting a change point focusing on some special distributions, normal distribution[3][4][5][6][7][8], for example, though, it does not fit the observed data in stock markets well under the circumstances of ex-kurtosis and heavy tails. Through our proposed approach, we can avoid using the conventional assumptions of the distributions for the observed data, such as, normal, log-normal, and non-Gaussian stable distributions etc, while detecting a structural change.

More precisely, We tackle the evolving values generated by a SDE as follows. First, we take account for the change sizes (over certain pre-defined levels) of the close values over the whole time span, as well as to measure the magnitudes of earthquakes, to regard these observed change sizes as occurred events, which follow some Poisson processes. Second, we adopt a Bayesian approach to detect the change of the intensity of a Poisson process to locate where a change point exactly lies at [9][10][11][12][14][15][16]. Hereafter, we just follow our previous research to display our proposed method [12].

3.1 Definition of change size and Bayesian inference

Following the above discussions, we propose to consider the change sizes of values over each time interval, and model the change sizes of values as a compound Poisson process, where an event happens when a change (jump) of the values exceeds a certain level, which is defined as some pre-decided value, we denote a change size of two continuous values by

$$\Delta V_t = V_t - V_{t^-}. \tag{11}$$

for a change size exceeds some pre-defined level, say, l, at time t, we define the event S_t as follows.

$$S_t = \begin{cases} 1, & |\Delta V_t| \geq l \text{ (event ocurrs)} \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

We then count all the events number $\mathrm{N}(\mathrm{S}_t)$ within time horizon $[0, T]$, and thus get a series of step signals, a series of events in the whole time span $[0, T]$, where $N(S_t) = \sum_{t=1}^{n} S_t$.

The general presentation of Bayesian inference can be shown as follows.

$$p(\beta|y) = \frac{p(y|\beta)p(\beta)}{p(y)} \tag{13}$$

where β is a set of parameters, and y is a set of given values where $\mathrm{p}(\beta)$ is the prior distribution for β, and $\mathrm{p}(y|\beta)$ is the likelihood, since

$$p(y) = \int p(y|\beta)p(\beta)d\beta \tag{14}$$

we then get the posterior distribution $\mathrm{p}(\beta|y)$,

$$p(\beta|y) \propto p(y|\beta)p(\beta). \tag{15}$$

Following the basics of the Bayesian approach, we can obtain S_t by the above-defined change sizes of the observations. Note S_t is a counting process of a set of time series data where each event occurs sequentially, namely, S_t, calculated by equations (11) and (12). Assuming that $N(S_t)$ follows two different poisson processes with two different intensities γ and δ divided by a change point k, we then have

$$N(S_t)|\gamma,\delta,k \sim \begin{cases} Po(\gamma), & \text{if } t \leqq k, \\ Po(\delta), & \text{if } t > k \end{cases} \tag{16}$$

So that we have $\beta = \{\gamma, \delta, k\}$ and $y = \{S_t\}$.

We further introduce the prior distributions for γ, δ, k as follows.

$$\gamma \sim \Gamma(\alpha_1, \alpha_2) \tag{17}$$

$$\delta \sim \Gamma(d_1, d_2) \tag{18}$$

$$k \sim U\{1, 2, ...T-1\} \tag{19}$$

Then, the likelihood function turns to be

$$L(\gamma,\delta,k) = \prod_{t=1}^{k} \frac{\gamma^{N(S_t)} exp(-\gamma)}{N(S_t)!} \prod_{t=k+1}^{T} \frac{\delta^{N(S_t)} exp(-\delta)}{N(S_t)!} \tag{20}$$

We thus get the following joint posterior distribution based upon the above prior distributions,

$$\begin{aligned} p(\gamma,\delta,k|N(S)) &\propto [\gamma^{\alpha_1-1} exp(-\frac{\gamma}{\alpha_2})\delta^{d_1-1} exp(-\frac{\delta}{d_2})] \\ &\times \prod_{t=1}^{k} \frac{\gamma^{N(S_t)} exp(-\gamma)}{N(S_t)!} \prod_{t=k+1}^{T} \frac{\delta^{N(S_t)} exp(-\delta)}{N(S_t)!} \end{aligned} \tag{21}$$

Assembling all the terms with γ, we have

$$p(\gamma|\delta,k,N(S)) \propto \gamma^{\alpha_1+(\sum_{t=1}^{k} y_t - 1} exp(-\gamma(\alpha_2^{-1}+n_k)) \tag{22}$$

where $n_k = k$. Clearly, it is the kernel of a Gamma distribution,

$$G(\alpha_1 + \sum_{t=1}^{k} N(S_t), (\alpha_2^{-1}+n_k)^{-1}) \tag{23}$$

Likewise, we get

$$p(\delta|\gamma,k,N(S)) \sim G(d_1 + \sum_{t=k+1}^{T} N(S_t), (d_2^{-1}+\tilde{n}_k)^{-1}) \tag{24}$$

where $\tilde{n}_k = T-(k+1)+1 = T-k$.

So that we have the conditional posterior of the change point k,

$$p(k|\gamma,\delta,N(S)) \propto \prod_{t=1}^{k} \frac{\gamma^{N(S_t)} exp(-\gamma)}{N(S_t)!} \prod_{t=k+1}^{T} \frac{\delta^{N(S_t)} exp(-\delta)}{N(S_t)!} \tag{25}$$

where $k = 1, 2, ..., T - 1$.

Having obtained all the conditionals, we therefore can carry out the MCMC (Markov Chain Monte Carlo Method) by using a Gibbs sampler.

4 Numerical experiments

In this section we apply our proposed approach to the simulated data to show how it works and also to check its performance. In the context, we first give the model setting in details. We then display some numerical examples on how to identify a structural change based upon our proposed methodology. Furthermore, we show the differences between incorporating a change point or not, concerning the VaR and asset allocation strategies.

4.1 Simulation studies

In this section, we carry out three numerical experiments using the artificial data, which are generated by the above discussed SDE systems with different parameters.

[Numerical experiment 1]

As shown above, we simulate the times when jump events occur, since those time intervals follow a exponential distribution, the events thus follow a Poisson process. Hereafter, we set the jump size J follows a normal distribution with mean μ_ξ and standard deviation σ_ξ.

Table 1 : Different λ in the same SDE

Parameters	μ	σ	λ	μ_ξ	σ_ξ	OBS
OBS:1 $\sim$ 1000	0.5	0.75	1.5	0	20	1000
OBS: 1001 $\sim$ 2000	0.5	0.75	5.5	0	20	1000

We generate 2000 values of the SDE. And the SDE has a structural change in the 1001st observation, namely, the first 1000 OBS are generated by λ =1.5, and the second 1000 OBS are generated by λ = 5.5. Except this different λ setting, the rest of the parameters are the same. We collect and count the event times over 10 points as a pre-defined interval. The parameter settings are summarized in Table 1.

And we run our Bayesian approach for this dataset, we then get the following results as shown in Figure 2.

The upper figure of Figure 2 shows the step function values where events occur. It can be observed clearly that the intensities of the step function values differ from the first half of the 2000 OBS to the latter half of the 2000 OBS, and the change point is around 1000. The lower one just shows the location of the change point we detect based upon our proposed Bayesian approach.

The identified change point is the 980th observation. There is a slight gap between the setting position 1001, though, it quite precisely catches the location of the change point and reflects that a structural change has occurred.

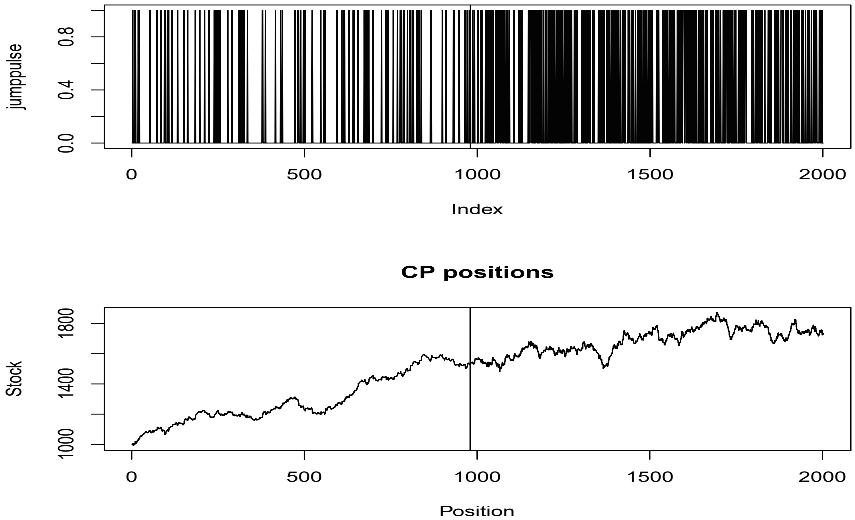

Figure 2 : Identified change-point location

[Numerical experiment 2]

We have generated artificial data for our 2nd numerical experiment, the parameters are summarized in Table 2.

Table 2 : Different λ and σ_ξ in the same SDE

Parameters	μ	σ	λ	μ_ξ	σ_ξ	OBS
OBS:1 $\sim$ 1000	0.5	0.75	1.5	0	20	1000
OBS: 1001 $\sim$ 2000	0.5	0.75	5.5	0	30	1000

2000 values are generated from the SDE. And the SDE has a structural change in the 1001st observation, namely, the first 1000 OBS are generated by $\lambda\ =\ 1.5$, and the second 1000 OBS are generated by $\lambda = 5.5$. Besides, the variance of the jump size σ_ξ takes different values. the rest of the parameters are the same. We collect and count the event times in 10 points as a pre-defined interval.

By running our proposed algorithm, we then get the following results as shown in Figure 3.

The upper figure of Figure 3 shows the step function values where the events occur.

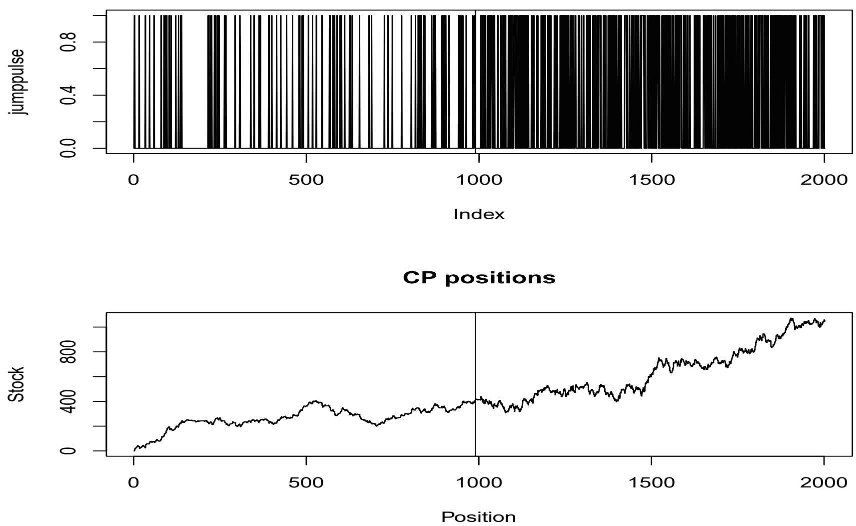

Figure 3 : Identified change-point location

It can be observed clearly that the intensities of the step function values differ from the first half of the 2000 OBS to the latter half of the 2000 OBS, and the change point is around 1000. The lower one just shows the location of the change point we detect based upon our proposed Bayesian approach.

The estimated location of the change point is at the 990th point, just as close as the setting change point position, the 1001st point. It still precisely catches the location of the change point and reports a structural change has occurred. Seen from Figure 3, the vertical line just flags where the change point lies at.

[Numerical Experiment 3]

2000 values are simulated by the SDE. And the SDE has a structural change in the 1001st observation, namely, the first 1000 OBS are generated by $\lambda = 1.5$, and the second 1000 OBS are generated by $\lambda = 5.5$. Besides, the diffusion term σ takes different values. We collect and count the event times over 10 points as a pre-defined interval. The parameter settings are summarized in Table 3.

By running our proposed approach, we then get the following results as shown in Figure 4.

The upper figure shows the step function values where jumps occur. It can be observed clearly that the intensities of the step function values differ from the first half of the 2000 OBS to the latter half of the 2000 OBS, and the change point is around 1000. The lower one just shows the location of the change point we detect based upon our proposed approach.

Table 3 : Different λ and σ in the same SDE

Parameters	μ	σ	λ	μ_ξ	σ_ξ	OBS
OBS:1 $\sim$ 1000	0.5	1.5	1.5	0	20	1000
OBS: 1001 $\sim$ 2000	0.5	0.75	5.5	0	20	1000

σ

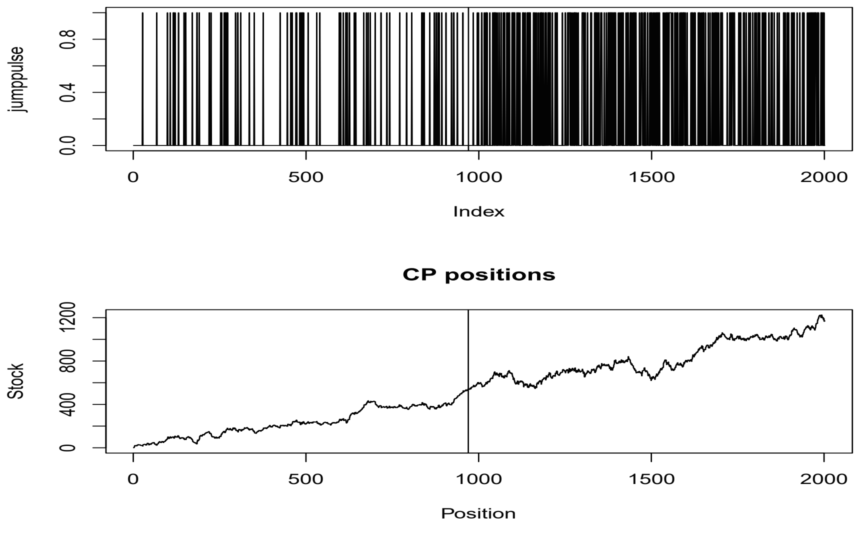

Figure 4 : Identified change-point location

The estimated location of change point is at the 970th observation, just as close as the setting change point, the 1001st observation. It still quite precisely catches the location of the change point and shows a structural change has occurred. Seen from Figure 4, the vertical line just flags where the change point lies at.

4.2 Impact of structural change on optimal portfolio

In this section, we investigate the impact of a structural change on an optimal portfolio. Simply, we utilize the simulated data in Example 3. Due to the change point, we have to reconsider the optinal ratios of the investing assets. First, we check the basic statistics between the structural change. Second, we investigate the volatilities of two segments divided by the change point. Third, we discuss the difference of the optimal weights of a portfolio between the two segments. And it is straightforward work to use the bisection search to detect the structural changes while several change points exist in the observe data.

Assuming that the simulated data in example 3 are the prices, we then have detected that the structural change occurred at the 970th point. The estimated means and variances of segments 1 and 2 are summarized in the following Table 4 and Table 5.

Table 4 : Estimated means and variances of segments 1 and 2

Segment	Mean	Variance
1	237.077	17807.41
2	812.733	31090.56
$1 \cup 2$	534.1255	107744.4

Table 5 : Quantiles of segments 1 and 2

Quantile	Segment 1	Segment 2
0.1%	3.737823	539.4399
0.5%	18.513673	546.3981
1%	19.795546	555.4929
2%	24.196148	561.5550
5%	33.860483	576.4283
10%	57.514599	601.4263
50%	220.93100	757.5230
99%	525.60411	1200.877

Seen from the tables, the means, variances, or quantiles are different among between the two segments, coming with the structural change. The mean of the segment 2 is 3 times as much as that of segment 1. And the variance of Segment 2 is about 1.7 times as much as that of segment 1. Furthermore, the mean and variance of the total data are between the segments 1 and 2.

Figure 5 shows the plots of the returns of segment 1 and segment 2, divided by the change point the 970th observation. We exclude the first value zero when we calculate the returns.

Figure 6 shows the estimated densities of the returns of segment 1 and segment 2, divided by the change point the 970th observation.

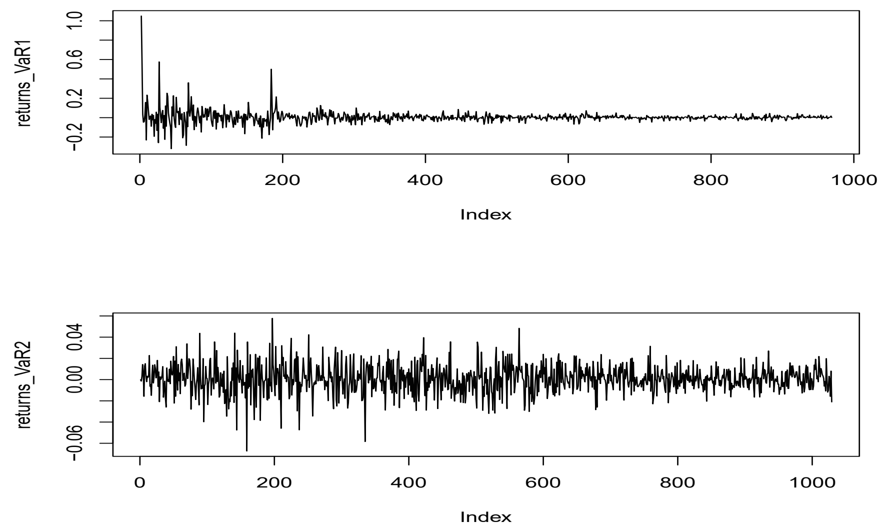

Figure 5 : Plots of Returns of Segments 1 and 2

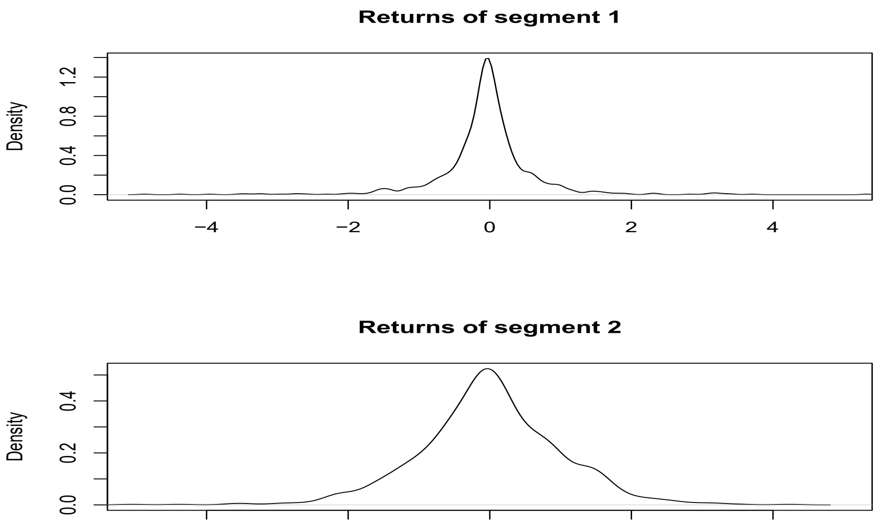

Figure 6 : Plots of standardized returns of segments 1 and 2

On the other hand, let us take a look at the statistical properties of the returns. These statistics are summarized in Tables 6, 7 and 8.

Seen from the tables, the statistical properties are much different after a structural change occurred

Table 6 : Statistical properties of each segment

Segment	Min	1st Qu	Median	Mean	3rd Qu	Max
1	-0.320600	-0.034810	0.008454	0.014100	0.049630	1.049000
2	-0.067320	-0.006659	0.000476	0.000750	0.008727	0.057790
$1 \cup 2$	-0.320600	-0.007966	0.001401	0.002859	0.011770	1.049000

Table 7 : Means and variances of returns of segments 1 and 2

Segment	Mean	Variance
1	0.00510922	0.004467167
2	0.00074975	0.000181651
$1 \cup 2$	0.002858924	0.002261407

Table 8 : Quantiles of returns of segments 1 and 2

Quantile	Segment 1	Segment 2
0.1%	-0.288167719	-0.0581049821
0.5%	-0.215520816	-0.0396203031
1.0%	-0.169976620	-0.0318643222
2.0%	-0.100615544	-0.0281510268
5.0%	-0.066196713	-0.0210443468
10%	-0.039176666	-0.0154860181
50%	0.002676573	0.0004755368
99%	0.218506220	0.0351478461

Suppose that we have a portfolio involved in several investing assets, and the returns (prices) of one asset combined in the portfolio have occurred a structural change, the statistical properties greatly changed as listed in Table 7 and Table 8. That means, the weights of the selected assets of a portfolio, such as based upon Markowitz mean-variance approach, should be adjusted as soon as a structural change is detected, so does the evaluation of the VaR.

Since the means, variances, and quantiles of each segment are greatly dispersed after the structural change. This situation should be concerned and the adjustment of the portions of the investing assets has to be done, according to Equation (9) (10).

Obviously, structural change introduces change point in the time series of the observed datasets. And it causes different statistical properties for the returns (prices) after the structural change.

Through the results of our numerical experiments, it has been confirmed that our proposed Bayesian approach can detect a structural change even in a SDE system, more importantly, the normal or other conventional assumptions of the observed data are not necessitated in our proposed approach to detect the structural changes or change points.

5 Concluding Remarks

We have proposed a Bayesian approach to detect structural changes, or change points in parameters of a SDE system. And we also have applied our proposed method to numerical experiments.

Through our numerical experiments, we have confirmed that our proposed Bayesian approach works well, especially under the circumstances that normality does not hold in the evolving time series. The estimated locations of change points of structural changes are close enough to the setting points. It means that our proposed Bayesian approach can detect where a change point lies at exaltly.

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